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A Limit Function for Equidistant Fourier Interpolation

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Denote by s_n the nth order Fourier polynomial of the odd function f of period 2π equal to 1 on $]0, \pi[$. The Gibbs phenomenon is caused by the well-known fact that

$$\lim_{n \to \infty} s_n\left(\frac{\pi x}{n}\right) = \int_0^{\pi x} \frac{\sin t}{t}\, dt.$$

An analogous Gibbs phenomenon is caused by a similar limiting behaviour of s_n^*, the nth order trigonometric polynomial interpolating f at $j\pi/n$ $(1 \leqslant j \leqslant 2n)$.

Let f be a periodic real-valued function on $\mathbb{R}$ with period 2π, of bounded variation on $[0, 2\pi]$, and with a jump discontinuity in $\xi \in [0, 2\pi]$. Let s_n^* be the nth order trigonometric polynomial of the form

$$s_n^*(x) = \frac{a_0^*}{2} + \sum_{k=1}^{n-1} (a_k^* \cos kx + b_k^* \sin kx) + \frac{a_n^*}{2} \cos nx \tag{1}$$

which interpolates f at the points

$$x_j = \frac{j\pi}{n}, \qquad 1 \leqslant j \leqslant 2n. \tag{2}$$

Let the index m be defined by

$$x_m \leqslant \xi < x_{m+1}. \tag{3}$$

As $n \to \infty$ the functions s_n^* exhibit a Gibbs phenomenon just as do the partial sums s_n of the Fourier series of f. This has been shown in [1] by investigating the behaviour of $s_n^*, s_n^{*\prime}$, and $s_n^{*\prime\prime}$ in the points

$$x_m + \frac{k\pi}{2n} \qquad (1 \leqslant k \leqslant 2n) \quad \text{as} \quad n \to \infty.$$

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The purpose of this note is to supplement these results by the following assertion. Without loss of generality suppose

$$f(x)=\begin{cases}1 & \text{for} \quad \xi<x<\xi+\pi\\ 0 & \text{for} \quad x=\xi,\ x=\xi+\pi\\ -1 & \text{for} \quad \xi-\pi<x<\xi.\end{cases} \tag{4}$$

THEOREM. *Suppose $x+\frac{1}{2}\notin\mathbb{Z}$. Then*

$$\lim_{\substack{n\to\infty\\ x_m\neq\xi}} s_n^*\left(x_m+\frac{\pi}{2n}+\frac{\pi x}{n}\right)$$

$$=\frac{\cos\pi x}{\pi}\cdot 8x\left\{\frac{1}{1-4x^2}-\sum_{k=1}^{\infty}\frac{16k}{[(4k-1)^2-4x^2][(4k+1)^2-4x^2]}\right\}, \tag{5}$$

$$\lim_{\substack{n\to\infty\\ x_m=\xi}} s_n^*\left(x_m+\frac{\pi}{2n}+\frac{\pi x}{n}\right)$$

$$=\frac{\cos\pi x}{\pi}\cdot 2\left\{\frac{1}{1-2x}-\sum_{k=1}^{\infty}\frac{64kx}{[(4k-1)^2-4x^2][(4k+1)^2-4x^2]}\right\}. \tag{6}$$

In (5) the index n runs through those values for which $x_m<\xi$; in particular, this condition is satisfied for every n if ξ is not a rational multiple of π. In (6) the index n runs through those values for which $x_m=\xi$. If $x+\frac{1}{2}\in\mathbb{Z}$ then $x_m+\pi/2n+\pi x/n$ coincides with one of the nodes and the limits in (5) and (6) coincide with the corresponding values of f.

The theorem establishes an analogy with the well known fact [3, II.9] that

$$\lim_{n\to\infty} s_n\left(\xi+\frac{\pi x}{n}\right)=\frac{2}{\pi}\int_0^{\pi x}\frac{\sin t}{t}\,dt. \tag{7}$$

The wave-shaped graphs of the functions defined by the limits (5), (6) and (7) respectively are responsible for the Gibbs phenomenon in case of Fourier interpolation resp. approximation.

For the proof note that for f as in (4) the function s_n^* is given by

$$s_n^*(y)=\begin{cases}\dfrac{(-1)^m}{n}\displaystyle\sum_{j=1}^{n^*}\frac{(-1)^j\sin ny}{\sin(y-x_{m+j})} & \text{if } n \text{ is even,}\\[2ex] \dfrac{(-1)^m}{n}\displaystyle\sum_{j=1}^{n^*}\frac{(-1)^j\sin ny}{\sin(y-x_{m+j})}\cdot\cos(y-x_{m+j}) & \text{if } n \text{ is odd}\end{cases} \tag{8}$$

[1] where

$$n^* = \begin{cases} n & \text{if} \quad x_m < \xi, \\ n-1 & \text{if} \quad x_m = \xi. \end{cases} \tag{9}$$

In order to simplify the proof we shall deal with the case of even n. Keeping in mind that $\cos y \to 1$ for $y \to 0$ the reader may check the validity of the arguments also for odd n. Putting $y = x_m + \pi/2n + \pi x/n$ in (8) we get

$$s_n^*\left(x_m + \frac{\pi}{2n} + \frac{\pi x}{n}\right) = \sin\left(x + \frac{1}{2}\right)\pi \cdot \frac{1}{n} \sum_{j=1}^{n^*} \frac{(-1)^{j+1}}{\sin\left(\dfrac{2j-1-2x}{2n}\pi\right)}. \tag{10}$$

LEMMA 1 [2]. *Let $\alpha \in]0, 1[$ be fixed. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=[\alpha n]}^{n-[\alpha n]} \frac{(-1)^{j+1}}{\sin\left(\dfrac{2j-1-2x}{2n}\pi\right)} = 0.$$

Proof. Since $1/\sin y\pi$ is continuous in $[\alpha, 1-\alpha]$, the limit above amounts to

$$\frac{1}{2\pi}\int_\alpha^{1-\alpha} \frac{dy}{\sin y\pi} - \frac{1}{2\pi}\int_\alpha^{1-\alpha} \frac{dy}{\sin y\pi} = 0. \quad \blacksquare$$

LEMMA 2. *Let the positive real numbers h and ε be given. Then there exists an $\alpha_0 > 0$ with the following property: for every $\alpha < \alpha_0$ there exists an index $N(\alpha, \varepsilon)$ such that for every $n > N(\alpha, \varepsilon)$ and for every choice of $[\alpha n]$ real numbers k_j satisfying*

$$\left.\begin{array}{l} 0 \leqslant k_j \leqslant 4\alpha n \\ k_j \neq h \end{array}\right\} \text{for } 1 \leqslant j \leqslant [\alpha n]$$

one has

$$\frac{1}{n}\sum_{j=1}^{[\alpha n]}\left(\frac{1}{\sin\dfrac{k_j - h}{2n}\pi} - \frac{1}{\sin\dfrac{k_j + h}{2n}\pi}\right) = \frac{1}{\pi}\sum_{j=1}^{[\alpha n]} \frac{4h}{k_j^2 - h^2} + \theta \qquad \text{where} \quad |\theta| < \varepsilon. \tag{11}$$

Proof. The Taylor series expansion of $\sin y$ furnishes

$$\lim_{y\to 0}\left(\frac{1}{\sin y}-\frac{1}{y}\right)\Big/ y=\frac{1}{6},$$

$$\frac{1}{\sin y}=\frac{1}{y}+\frac{y}{6}+o(y) \qquad \text{as} \quad y\to 0.$$

Choose $\alpha_0\in\,]0, \varepsilon/2\pi[$ such that $|o(y)|<\varepsilon/3$ as soon as $|y|<2\alpha_0\pi$. Then for $n>h/4(\alpha_0-\alpha)=N(\alpha,\varepsilon)$ one has, uniformly for $1\leqslant j\leqslant[\alpha n]$,

$$\frac{1}{\sin\dfrac{k_j-h}{2n}\pi}-\frac{1}{\sin\dfrac{k_j+h}{2n}\pi}=\frac{2n}{\pi(k_j-h)}-\frac{2n}{\pi(k_j+h)}-\frac{h\pi}{6n}+\theta_1$$

where $|\theta_1|<2\varepsilon/3$ and $h\pi/6n<\varepsilon/3$. This implies (11). ∎

Proof of the Theorem. Consider first the case $x_m<\xi$ (3) which implies $n^*=n$ (9). It suffices to consider the factor of $\sin(x+\frac{1}{2})\pi$ in (10). Given any $\alpha\in\,]0, 1[$ let $n_1=[\alpha n]$. By Lemma 1 we have

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\frac{(-1)^{j+1}}{\sin\dfrac{2j-1-2x}{2n}\pi}$$

$$=\lim_{n\to\infty}\frac{1}{n}\left\{\sum_{j=1}^{n_1}\frac{(-1)^{j+1}}{\sin\dfrac{2j-1-2x}{2n}\pi}+\sum_{j=n-n_1+1}^{n}\frac{(-1)^{j+1}}{\sin\dfrac{2j-1-2x}{2n}\pi}\right\}$$

$$\lim_{n\to\infty}\frac{1}{n}\left\{\sum_{j=1}^{n_1}\frac{(-1)^{j+1}}{\sin\dfrac{2j-1-2x}{2n}\pi}+\sum_{j=n-n_1+1}^{n}\frac{(-1)^{j+1}}{\sin\dfrac{2x+2(n-j)+1}{2n}\pi}\right\}$$

$$=\lim_{n\to\infty}\frac{1}{n}\left\{\sum_{j=1}^{n_1}\frac{(-1)^{j+1}}{\sin\dfrac{2j-1-2x}{2n}\pi}+\sum_{j=0}^{n_1-1}\frac{(-1)^{j+1}}{\sin\dfrac{2x+2j+1}{2n}\pi}\right\}$$

$$=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n_1}(-1)^{j+1}\left\{\frac{1}{\sin\dfrac{2j-1-2x}{2n}\pi}-\frac{1}{\sin\dfrac{2j-1+2x}{2n}\pi}\right\} \tag{12}$$

We now apply Lemma 2 separately to the two sums in which j respectively runs through the odd and even values. If first n tends to ∞ and then α

together with ε tend to zero, then the last limit above turns out to be equal to

$$\frac{8x}{\pi}\sum_{j=1}^{\infty}(-1)^{j+1}\frac{1}{(2j-1)^2-4x^2}$$
$$=\frac{8x}{\pi}\left\{\frac{1}{1-4x^2}+\sum_{k=1}^{\infty}\left[\frac{1}{(4k+1)^2-4x^2}-\frac{1}{(4k-1)^2-4x^2}\right]\right\}$$
$$=\frac{8x}{\pi}\left\{\frac{1}{1-4x^2}-\sum_{k=1}^{\infty}\frac{16k}{[(4k+1)^2-4x^2][(4k-1)^2-4x^2]}\right\}.$$

If $x_m=\xi$ (3) then $n^*=n-1$ (9) and in (12) the term for $j=0$ has to be omitted. In the limit itself this amounts to an addition of $(2/\pi)\cdot 1/(1+2x)$ which results in formula (6).

Remark 1. If the definition of f as in (4) is changed by putting $f(\xi)=c$ (and for convenience $f(\xi+\pi)=-c$), then for $x_m=\xi$ in (12) the term for $j=0$ has to be replaced by its $(-c)$-fold. In the limit itself this amounts to an addition of $(2/\pi)\cdot(1+c)/(1+2x)$ which results in the formula

$$\lim_{n\to\infty} s_n^*\left(x_n+\frac{\pi}{2n}+\frac{\pi x}{n}\right)$$
$$=\frac{\sin(x+\frac{1}{2})\pi}{\pi}\cdot 2\left\{\frac{1}{1-2x}+\frac{c}{1+2x}\right.$$
$$\left.-\sum_{k=1}^{\infty}\frac{64kx}{[(4k-1)^2-4x^2][(4k+1)^2-4x^2]}\right\}.$$

Remark 2. In order to compute the limit functions up to an error $<\varepsilon$ the series may be approximated by a partial sum of at least $K=\frac{1}{4}(\sqrt{32/\pi\varepsilon+4x^2}+5)$ terms. (If $\varepsilon<8/\pi x^2$ then the error may conveniently be majorized by integrals from $K-1$ to ∞.) On the interval $[0, 10]$ the values of the functions (5) and (6) may be computed with an error smaller than $\varepsilon=10^{-5}$ by adding up 255 terms of the series. In agreement with the assertions in [1] on the limiting behaviour of s_n^* for $n\to\infty$ the maxima computed in this way are

$$1,28228\ldots \quad \text{for} \quad x=0,917\ldots \text{ in (5)},$$
$$1,06578\ldots \quad \text{for} \quad x=0,877\ldots \text{ in (6)},$$

while the function (7), as is well known, obtains its maximum

$$1,17898\ldots \quad \text{for} \quad x=1.$$

Remark 3. The main idea of this note is already contained in the following result of de la Vallée-Poussin [2] [4, X example 11].

Suppose

$$x_m < \xi < x_{m+1}$$

and let

$$\theta_n = \frac{n(\xi - x_m)}{\pi},$$

i.e.,

$$\xi = x_m + \frac{\theta_n \pi}{n}.$$

Define for $0 < \theta < 1$

$$\psi(\theta) = \frac{\sin \pi\theta}{\pi} \sum_{j=0}^{\infty} \frac{(-1)^j}{\theta + j} = \frac{\sin \pi\theta}{\pi} \int_0^1 \frac{t^{\theta - 1}}{1 + t} \, dt \tag{13}$$

Then

$$\lim_{n \to \infty} \{s_n^*(\xi) - [\psi(\theta_n) f^-(\xi) + \psi(1 - \theta_n) f^+(\xi)]\} = 0. \tag{14}$$

In fact, in the same way as in the proof of the theorem one obtains for any $x \notin \mathbb{Z}$

$$\begin{aligned} \lim_{n \to \infty} s_n^* \left(x_m + \frac{\pi x}{n}\right) &= \frac{\sin \pi x}{\pi} \left\{ \sum_{j=1}^{\infty} \frac{(-1)^j}{x - j} + \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{x + j} \right\} \\ &= \frac{\sin \pi x}{\pi} \left\{ \sum_{j=0}^{\infty} \frac{(-1)^j}{1 - x + j} - \sum_{j=0}^{\infty} \frac{(-1)^j}{x + j} \right\} \\ &= \psi(1 - x) - \psi(x). \end{aligned} \tag{15}$$

This agrees with (14) but interprets this formula also for values of x off the jump abscissa. At the same time this shows that for $0 < x < 1$ (i.e., between x_m and x_{m+1}) the limit function (15) may be written in the form

$$\lim_{n \to \infty} s_n^* \left(x_m + \frac{\pi x}{n}\right) = \frac{\sin \pi x}{\pi} \int_0^1 \frac{t^{-x} - t^{x-1}}{1 + t} \, dt,$$

As a consequence, for $x_m < \xi$ one has

$$\begin{aligned}\lim_{n\to 0} s_n^*\left(x_m+\frac{\pi}{2n}+\frac{\pi x}{n}\right) &= \psi\left(\frac{1}{2}-x\right)-\psi\left(\frac{1}{2}+x\right)\\ &= \frac{\cos \pi x}{\pi}\int_0^1 \frac{t^{-1/2-x}-t^{-1/2+x}}{1+t}\,dt \qquad \left(-\frac{1}{2}<x<\frac{1}{2}\right).\end{aligned}$$

Remark 4. Formula (15)—and therefore also (5) and (6)—may also be derived from the interpolation formula

$$s_n^*(y)=\frac{1}{n}\sin n(y-x_k)\sum_{j=-\infty}^{+\infty}\frac{(-1)^j f(x_{k+j})}{y-x_{k+j}} \tag{16}$$

[2] [4, X example 1] applied to f as given in (4), where s_n^* is as in (1) and x_j is as in (2). For $y=x_m+\pi x/n$ and $x_k=x_m$ formula (16) may be written as

$$s_n^*\left(x_m+\frac{\pi x}{n}\right)=\frac{\sin \pi x}{\pi}\sum_{j=-\infty}^{+\infty}\frac{(-1)^j(-1)^{[(j-1)/n]}}{x-j}.$$

Taking into account that the function $1/(x+y)$ is ultimately decreasing and convex as a function of y for $y\to\infty$ one obtaines

$$s_n^*\left(x_m+\frac{\pi x}{n}\right)=\frac{\sin \pi x}{\pi}\left(\sum_{j=1}^{n}\frac{(-1)^j}{x-j}-\sum_{j=0}^{n-1}\frac{(-1)^j}{x+j}\right)+O\left(\frac{1}{n}\right)$$

as $n\to\infty$. This agrees with (15).

Remark 5. Recall that the digamma function $\Psi(x)$ is defined by

$$\Psi(x)=\frac{d\log\Gamma(x)}{dx}=\frac{\Gamma'(x)}{\Gamma(x)}=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+x}\right)-C-\frac{1}{x}$$

where C is Euler's constant. By (13) one has

$$\psi(x)=\frac{\sin \pi x}{\pi}\left[\Psi\left(\frac{x+1}{2}\right)-\Psi\left(\frac{x}{2}\right)\right].$$

By formula (15) this allows to express the limit function in terms of the gamma function. This observation is due to N. Ortner and P. Wagner.

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